Psycho-Mathematical-Ontology

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Abstract

Does cognitive science have something to teach us about the fundamental structure of mathematical reality? This paper puts forward a cautious yet firm answer: a lot. It does so on two levels.

The first is a philosophically conservative, methodological one. It claims for the relevancy of particular portions of cognitive knowledge to the exploration of mathematical ontology, classically conceived. The point will be demonstrated using a few selected historical case-studies.

The second, much more philosophically ambitious level, suggestively 'applies' the cognitive revolution to the philosophy of mathematics, with the mainstream, *logical* foundations taking the role of behaviorism in this analogy. The application results in a call for an alternative, *cognitive* foundation for mathematical ontology. Such foundations could hardly even be sketched here, but I will address central motivations for it, as well as the main challenges that stand before any such attempt.

Introduction

The Nature of Mathematics

In the foundational, layered picture of the sciences, mathematics supports it all – in particular by underlying physics. Physical theories may change, not to mention higher-level chemical, biological, neural or psychological ones. "New" mathematics might have to be "developed" alongside; but it is always actually *discovered*. And once discovered, it might turn out to be just as useful for a new application it wasn't developed for. As a practice, it is a monotonically growing endeavor: once something is proved, it will never be disproved; only strengthened or generalized. It is thus non-contingent. And independent of higher-level, scientific revisions.

This is pretty much the standard picture; any approach that disputes any of it, should be a priori suspect of not be dealing anymore with mathematics as such. My intention here is to approach this more carefully, discerning between that 'nature' of mathematics and actual exploration of the mathematical universe.

[1] Updated version can be found at http://huji.academia.edu/AvivKeren/Papers

Logical Foundations – a Crash-Course

The century-old, mainstream approach to the foundations of mathematics rests on logic, designed expressly to take psychology completely out of the picture. (In fact, Frege's and Husserl's famous arguments that formulate 'Psychologism' and put it forward as a sin stem from here). The ontology itself is still Platonistically conceived: as existent as atoms are, but abstract, autonomous of time & space; and completely independent of our language, thought, and practices. It is grounded, however, onto language – albeit a 'non-human', formal one. There is some metaphysical connection between mathematical *ontology* and propositional *facts*, and it is the latter with which the mathematician interacts – through axioms and theorems, that span theories.

Essentially, the mathematician's work is to prove theorems, based on previous ones and ultimately on a basic set of axioms; all else – visual illustrations & perceptions, intuitions, etc. – is beside the (foundational) point, a heuristic aid of no philosophical, metaphysical, importance. It is mathematics that's being done if and only if this set of axioms is consistent (or in Frege's Logicism, true). Ideally, that consistency would itself be *proved* (or in Frege's case, the axioms would be self-evident, to an extent that somehow transcends psychology-level assurance).

This ideal of Hilbert (who set the mathematicians' attitude for the rest of the 20th century), however, has been shown (by Gödel, mostly) to be unattainable (Frege's has collapsed, too). On the more 'technical' front, consistency cannot be proved by simpler, more trustworthy means, but only stronger ones (which makes as much sense as paying back a loan through taking another one, of a higher interest-rate). On the conceptual front, it turns out that the mystical connection between a consistent theory and the structure behind it is not at all as mathematically straight-forward as Hilbert would have it. Neither existence nor uniqueness are granted: A consistent set of axioms may conceivably be satisfiable by *no* structure[2]. Or it may be satisfiable by a few essentially different ones, that are, however, indistinguishable in terms of the theory's fixed language. More concretely put: even if we had in our possession a reasonable set of axioms sufficient to describe, say, number-theory (and, by Gödel's incompleteness theorem, we can't), it would not be enough to pin-point the natural numbers themselves (however we come to know this most elementary mathematical structure, it is not through axiomatic definition, which is the only method the logical foundations make room for).

Despite the *philosophical* failure of the logical foundations, the approach formed upon them is accepted by the mathematical community: Rigour is *logical* rigour. As long as the mathematical work can be ideally brought down to a formalized version that relies upon consistent axioms, this is meaningful mathematics. And even though the consistency cannot ordinarily be proved, it is enough to be 'as consistent as' ZFC. This is a set theory that is accepted as valid by most mathematicians, yet rich with idealizations that are enough to, in a sense, contain ordinary mathematical ontology, while strong enough to prove most known mathematical theorems. Just to give a taste of the richness of the ontology it commits to:

[2] This turns out to depend on the expressive strength of the logic. But even if one kneels before first-order logic, the fact that this requires a (non-trivial) proof, is the philosophical point here.

according to ZFC, there exist more real numbers (different lengths) than could possibly be described, named, or otherwise cognitively grasped (assuming bounds on language & cognition that are generally accepted). As long as only the *theory* of the mathematical structure of the real numbers has to be able to be brought down into a linguistic form we can epistemically relate to (while the elements themselves can stay 'up there'), this poses no problem – as the logical foundations would have it.

This approach formed at the beginning of the 20th century, in a suitable intellectual and scientific climate. It developed immensely – technically, as an independent mathematical branch on its own; but without regard to scientific advancements external to mathematics.

Meanwhile, the cognitive revolution was taking place.

The Breadth of the Unconscious

A methodological shock to the age-old philosophical imperative to "know thyself", is the modern understanding that almost all that is going on under our hood, takes place not only subconsciously, but completely out of introspection's reach. That the things that are the most important or frequent are cognitively automated, hidden from consciousness, so as to not consume its precious, (surprisingly) extremely limited resources. And that all this concerns even the most seemingly-conscious of phenomena (such as our linguistic capacity). That perception can be not only highly partial (compared to what's really going on), but *systematically* deceived. That, more specifically, phenomenological simplicity, seemingly trivial cognitive complexity, can be – philosophically – disastrously misleading.

Part I – Methodology

Intuitions, introspection, and even perception, play a central role in the mathematicians' work, and specifically in the definitions they accept, explore & alter; in determining the specific objects or subject-matters as they agree upon them and understand them. These previous perils are especially acute when it comes to mathematics, with its ideally-absolute standards of explicitness, clarity and non-contingency. Is mathematics exempt from them?

Even if we fully adopt an underlying Platonist philosophy (which I will not be disputing in this section), to modern, informed eyes, Frege's trust in self-evident basic truths seems wholly unwarranted (as he has notoriously learned the hard way). As is Gödel's later hope (responding to his own proofs concerning the limits of logic) for some form of mathematical 'perception' (akin to our perception of *physical* reality), that can straight-forwardly be trusted.

Up until now, the main rigorous guiding restraint mathematicians had (or could agree upon) was logic – clarifications or revisions brought about in response to blunt contradictions. Yet there is no orderly, algorithmic way to verify that there are none. Thus, even the devout Hilbertian follower who views ontology through the (limiting) prism of consistency, ought to embrace any help she can get in the mathematical endeavor, which in an essential way involves forming & formalizing intuitions & new concepts.

This isn't a big issue in practice, however, as most mathematical work is done in realms which are considered to be safe from inconsistencies. But much more profoundly than that, even if we do formally accept all that is consistent as 'legitimate mathematics': mathematics as the endeavor, the activity that it actually is, has a much finer structure than just *random valid deductions from arbitrary consistent sets of axioms*. Mere logical correctness doesn't make for good mathematics. What makes concepts, definitions, specific subject-matters & structures, theorems, conjectures etc. 'important' or 'interesting'[3], it is in general hard, if not impossible, to determine directly in relation to the logical level (though one can try to define, say, 'important' definitions as ones that enable proving or shortening proofs of many 'important' theorems, etc.). And even if is ultimately *possible* to be put in such terms, there seem to be little reason to insist on doing so but 'logical' dogmatism. The preliminary anti-psychologistic concerns simply do not apply to this finer structure of mathematical reality (having no quarrel with the aforementioned, *higher-level*, 'nature of mathematics').

My contention here is that 'cognitively-informed awareness' can shed light on unquestionably-mathematical matters (as opposed to just "the psychology of mathematics"), and hopefully contribute to the field itself. One must bear in mind Kripke's point, that higher-level, more 'theoretical' truths, can be found through lower-level, more 'empirical' methodology. This is much more general than just computers proving mathematical theorems. For example, biological research can figure out abstract engineering principles (e.g. how to build a functional eye), and even higher-level relations (e.g. dynamically, evolutionarily, there are many continuous routs converging to a functional eye).

The best demonstration of that contention would undoubtedly be, to apply this ideology and reach an actual mathematical achievement (the way computerized proofs showed their value by answering open conjectures). This, however, would require getting into a deep, lively mathematical area, and mostly, is simply damn hard to do. I shall instead sketch here three *historical* cases where such a view *could* have moderated the surprise (a testimony of better philosophical as well as practical understanding), and perhaps help guide the development.

The Comprehension Axiom

A first course on set-theory is ordinarily taught (, as it should be,) as "naïve set theory": without any axiomatic foundation; no definition of what sets are, what sets there are, or what sets there aren't. Particular sets & operations on sets are introduced casually as needed. Students form some intuitive conception of what they can & cannot do, and the 'philosophical' issue never stands in their way of mastering the course's material. Even after doing so, the so-called 'axiom of comprehension', when suggested, may seem not only perfectly plausible, but as the precise formalization of the rule that has actually been used to introduce sets & operations all along:

For every property $p(x)$, there is the set $\{x \mid p(x)\}$ (of all the sets of which p holds).

[3] Not to mention the role of graphic illustrations, diagrams and the likes.

Imagine (or hopefully, recall) the shock when Russell's $\{x \mid x \square x\}$ is shown to quickly lead to paradox.[4]

The mathematical community's alternative, more cumbersome suggestions to what the laws of set-formation may be aside, let us take a cognitively-modern look at what *might* have been going on here.

Set-Theory in particular introduces idealizations that take mathematics beyond its classic, more computational origins, but let's put the horses ahead of the cart; let's consider computation as being extended into set-theory (as the former is more 'concrete') rather than the latter as foundations for the former (as set-theory is richer, powerful enough to embed everything else within it). 'Programming' set-theory would involve operations that check whether an element does not already belong to a set before adding it, etc. Things get more complex as we might also have to process processes (we never have, for example, the complete whole of the natural numbers to collect together); but in any case, there should be some constructive version of set-theory that is underlying what's going on in our mind throughout the course. The possibility that some such theory is the "true" theory, with which, also, the mathematician's cognition is correctly 'in sync', will suffice for my point. (Interestingly, all Post-Russell's-Paradox suggested approaches went some distance towards constructivity; this – despite set-theory's Platonistic, anti-constructivist primal nature.)

Now: These constructive procedures, that operate directly on the ontology itself (various elements), seldom have our attention directed towards them, not to mention being expressed linguistically (and heaven forbid, formally). As they are being ran over & over again, they disappear into automaticity; the more essential, the more frequent – the more invisible they become. And this may certainly be expected to include a vast apparatus of operations already in place, that serves us in our daily lives since childhood. As with Piaget, who famously found logical abilities that appear at specific ages in early childhood – a long way from the grown-up now studying set theory.

The linguistic level is *not* where the most *elementary* operations are reflected; not naturally, not intuitively. Rather, it reflects the *conscious*, highest level, referring to the most overarching complexes of operations. And as is well known by now (e.g. through visual illusions), *this conscious level is not at all a mathematically-exact generalization* of the complex inner-operation. It is but a *heuristic approximation*. An approximation that may be refined in response to experience, to counter-examples. But it is rarely fully-detailed, *and never with any subjective yet valid assurance of being so.*

The easy, skeptical conclusion from all of this, is that we needn't have waited for an obvious paradox such as Russell's to come up in order suspect the faulty axiom. The intuitiveness of it may spring from other sources than just pure-and-simple mathematical truth, sources of cognitive efficiency with dubious mathematical legitimacy. What might be going on here, then? Let's be wildely speculative:

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[4] The regular intuitive response is that "no set can contain itself". It is important to note here: not only does this *not* resolve set-theory's paradoxes, but a set theory can actually allow for such circularity while keeping to consistency!

Consider an analogy with visual attention & perception. Phenomenologically, it feels as if we perfectly grasp our whole view, everything that is physically available to us. We wouldn't, for example, miss a big, unrelated gorilla walking right through the middle of the scene we are focusing on. Yet cognitive science teaches us that this is far from being the case. Conscious attention, working memory etc. are extremely restricted; our glance's spotlight very focalized. Many sub-conscious procedures are doing almost all of the work, bringing to our attention whatever needs to be brought there, depending crucially on context in a top-down manner that lets us focus on what is relevant to our goals. Moreover, they *cheat* us at the detailed level in order to *approximately* show us the big picture (equating colors that are actually different, guessing underdetermined 3d-structures, etc.). Overall, while reality as physically perceived is built bottom-up (from the tiniest visual atoms, to edges, to objects, etc.), the constructive cognition that follows its contours lays hidden, and our conscious perception is reversed.

So, can something related be happening in our perception of set-theory? Our linguistic-level, conscious intuition, which phenomenologically seems to be *constitutive* of the reality of sets, may have actually been following complex, concrete constructions that have been going on throughout the course, summarizing it efficiently into syntactic form: "Whatever you may say: if, formally, it may be a set – then it **is** a set". This has certainly been true throughout the course – *only the causal direction has been reversed*. And this reversal alters the scope of the comprehension axiom's validity: From applying to all already-*constructed* sets, it is applied to all linguistic properties. This reversal is devastating:

Each comprehension instance encountered (throughout the course) was made valid *within a specific context*; it followed a concrete set already constructed. Our deceitful cognitive tendency to de-contextualize, to holistically, simplistically grasp reality as a whole, unconscious of the many procedures that, making use of extremely focused physical perceptions and conscious resources, bring that holism into being – this tendency, here in this suggested analogy, turns into the naïve belief that we can grasp and operate on the world of *all sets* as a whole. Then, any property, by its very nature, bisects this mystical, unconstructed universe of sets, defining a legitimate, well-behaved set. (This is indeed 'the fault' most non-naïve set-theories single out and prohibit. But the available justifications are of a more pragmatic nature – "no new paradoxes have been found, and we found ways to recover enough of set-theory that is sufficient for doing mathematics". And there is no guided, methodological way to look for other possible paradoxes, and correctly delimit the universe further.)[5]

Treating language as the foundational level rather than just a means to *refer* to actual reality (the way it is treated outside mathematics), brings in a discrepancy between the two. As it so happens, this discrepancy resulted in an actual logical contradiction that luckily was found. But even had there been **no** contradiction, we would still be miss-representing actual mathematical reality. Yet in order to say such a thing, we must drive a deeper wedge between logic and mathematical ontology, and this shall be the concern of Part II.

[5] This is not to deny that the mathematical solution is what inspires & directs my analogy to begin with.

Whatever the correct account of the full cognitive & mathematical details may be, the mere reasonableness of such a possibility is my point. What we would end up with is thus a conspicuous example of how the mathematicians' unawareness of their own internal processes may lead to amazingly robust – yet demonstrably false – fundamental intuitions.

Ordinals vs. Cardinals

(This case-study and the next are somewhat more technically involved, and so I will make do with only a broad overview here.)

[Demonstrates how non-essential facts can sneak in subconsciously, distorting our theorizing, even when done into 'rigorous' first-order logic. To Be Completed.]

Compactness

[Awareness of the central role of cognition in the mathematician's work turns a specific focus on the centrality of finiteness (as an obvious cognitive constraint) – even when dealing with the actual infinite. Motivating a search for underlying finiteness, this may shed light on and actually guide the development of mathematics itself. T.B.C.]

Part II – Cognitive Foundations

Philosophical advancements can come about by ways other than a succinct knock-down argument or a dogma-rebutting thought experiment. The cognitive revolution, beyond its empirical body of knowledge, is one prime such example, resulting instead from interaction with scientific progress in various independent but related areas.

Accepting its basics as fundamentally correct, and extending its reach in a natural, analogical manner to the realm of mathematics (natural from the cognitive scientist's point-of-view), results in what I shall term "a cognitive foundation of mathematical ontology". This basic narrative is what this part aims to set out. Such a foundation, too, will not be campaigned–for on the basis of some direct, clear-cut, purely-philosophical concern; rather, the analogy itself will carry some weight – to the extent that it is substantial.

The many intertwined issues in the philosophy of mathematics will not be addressed in this paper (despite being a major part of the project at large). The central challenge here is in bringing the (relatively) old news of the cognitive revolution to the understanding of mathematical ontology – *while respecting the unique nature of the realm*.

The Logical Foundations as Behaviorism

A precursor to Positivism, the logical foundation approach was very much in its spirit. With the mathematical universe being distinct from the physical universe, a careful, metaphysically-modest approach that lives up to (those days') 'scientific' standards had to make do with its manifestation in the physical world. While mathematics apparently 'takes place' in physical world, it *applies* to it, seeming to be epistemically prior, a prerequisite for exploring the latter; thus offering no obvious way to understand mathematics *through*, based on, understanding the physical. We are left with the traces of mathematics' manifestation in humans. With introspection and mental states & reality positivistically banned, and visual materializations (diagrams, sketches, etc.) considered as sporadic heuristic aids unfit to serve as a full foundation, mathematics' linguistic expression takes the front. All else is beside the point; mathematical reality was now to be "operationally-defined".

Logic may indeed seem a reasonable candidate for mathematics to lay on. Conceivably, there may not be any more to mathematical objects then the relations between them. And if a finitary linguistic description of a structure could define it and give out all its laws, that could be an impressive account addressing the age-old challenge, of how us, finitary beings, relate to infinitary structures. However, we know now that under reasonable constraints, this is unachievable. A seemingly elementary structure such as the natural numbers is underspecified by its theory, which in turn is underspecified by a reasonable set of axioms (reminiscent of Chomsky's "poverty of stimulus" argument – only tightly proved).

But these failures in Logicism & Formalism's own internal terms aside, in modern eyes, 'behavioristically' grounding mathematical reality onto logic may seem not only unjustified, but overly-restrictive for the exploration of ontology itself.

The Cognitive Revolution as a Philosophical one

The breadth of the non-conscious and its illusive nature, rather than just an empirical finding, brings with it a theoretical revolution in what it means to account for a phenomenon in these areas, to really understand, explain it. To describe an actual mechanism that gives rise, or at least *can* give rise, to a phenomenon, can be very different from listing the *laws* that govern it. One can certainly have the latter without the former. And vice versa – even though the mechanism metaphysically determines the laws.

Rather than a methodological matter of "simply approaching" the same phenomenon through its mechanism rather than its laws, this shift comes with the legitimatization of "new" types of theoretical entities – *mental states* (or rather, rehabilitation of obvious folk-psychological ontology). Entities that now become a central subject of research for their own sake. (Without getting here into the origins, motivations and advantages of this dramatic change within psychology and how relevant it is to mathematics, I endorse the analogous shift of focus towards a direct, non-linguistically-mediated focus on mathematical ontology.)

The conscious part of the mental is much diminished (within the extended picture now also including The Breadth of the Unconscious); and it is certainly not foundational, as in phenomenology. However, it, too, is rehabilitated (from the devastating behavioristic era). Methodologically, it is nowadays successfully used as a heuristic guide in cognitive research. And more importantly: Theoretically, it is also part of what a cognitive account must ultimately explain.

Moreover, with the explosive development and discovery of the general power of computation (going way beyond classical conceptions of 'calculation'), the possibility of grounding the mental world in the physical one becomes a reality. And turns into a working-philosophy.

((Greenwood, 1999), (Piccinini, forthcoming), (Chalmers, 1994))

The Cognitive Revolution "on" the Foundations of Mathematics

I now turn to the "application" (as I see it) of the cognitive revolution (particularly its theoretical 'gist' as presented above) to the philosophy and foundations of mathematics. Beginning with a metaphysical parsimony, a naturalism taking all non-physical to be mental, in turn reducible to the physical (at least in the functionalist sense of abstract mechanisms being realized by physical reality)[6]:

No doubt mathematicians have some level of cognitive 'representation'[7] of the ontology they deal with (the objects, as well as the relations between them and the whole structures). *The presupposition is that (at least an idealized version of) such representation can be formalized to mathematical precision.* Building on that, the central hypothesis is this:

No further metaphysics is necessary; no external, independent reality – this *is* the mathematical reality. The cognitive representation level can successfully serve as a foundational basis for the whole of mathematics' ontology.[8]

This does not – contra most similar approaches – mean a straight-up denial of mathematicians' commonplace fundamental intuitions. The realistic talk, which refers to an autonomous reality, relative to which the sentences uttered are rendered true or false (proofs serving as mere witnesses)[9] – has to be accounted for, ultimately.

In general: Russell's philosophical claim that mathematics can be reduced to logic necessitated a large body of technical, non-philosophical work to support it. The same goes for the central hypothesis and the project it springs – only sevenfold. I now address a few philosophical issues that are central to such a project.

A Cognitive Basis without Contingency?

Up front, the mission may seem contradictory:

Cognitive science, being a science, concerns the acquisition of empirical knowledge, of a contingent nature; meanwhile, mathematicians' conception of the mathematical universe is of a timeless, abstract, non-human, a-priori-explorable, non-contingent one. Yet *a foundation needs to be more 'fundamental' than what it gives foundations for*.

Part I already suggested that a scientific methodology, and certainly scientifically-lead philosophical overturns, do not necessarily bring out only contingent facts. But the challenge here is metaphysically deeper. Cognition itself must have a non-contingent component, if it is to underlie mathematics *as such*; all 'human cognition' can support is 'human mathematics'. Luckily, there is such a basis for theoretical, general cognition:

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[6] Only with the emergence of modern, general computation and the unfolding of its power, did the possibility of the mental realm being animated through the material become conceivable *as a practical research programme*.

[7] 'representation' being a term that fits a Platonistic conception.

[8] I am thus committing – regarding mathematics! – to a sort of *semantic internalism*.

[9] When it comes to mathematics, just as in cognitive science at large: Language's relation to non-linguistic cognition and to objective reality should be cognitively accounted for. At least as a default assumption, it should be theoretically grasped uniformly over mathematics and the rest; otherwise – the differences have to also be cognitively accounted for.

computationalism. Already abstracted away to its non-physical core, a computational ontology squares with mathematical ontology in most metaphysical respects. A suggested unification, of mathematical with *abstract* mental ontology, can do so independently of classical problems concerning the relation between cognition and its abstract modeling (which are left to philosophers of mind). The challenge thus becomes metaphysically more modest, leaving primarily the gap between computational and full-blown mathematical ontology to be dealt with.

This is the importance of the aforementioned presupposition.

Ontological Rigour

A well-defined standard for mechanistic accounts of mathematical ontology determines a notion of *ontological rigour*, of objects and structures being given such an account. Intuitively, the ability to give such an account for an object means that we fully understand what we are talking about, in a sense that the familiar definitional standard does not capture. While a difficulty to give such an account, no obvious (possibly-)underlying cognitive processes having been made explicit, means that the object has to be better-understood, and perhaps reinterpreted.

Essential Finitism

Ontologically-rigourizing mathematics onto a cognitive basis requires that idealizations themselves be mechanistically accounted for, reduced. The number-one challenge here is that any mechanism must be finite. Thus, by supposition (the central hypothesis), any infinite mathematical object has to be reinterpreted so as to reveal the way, the sense, in which it is actually finite. Or else – be deemed ("cognitively") meaningless. The metaphysical parsimony thereby turns the age-old *epistemic riddle* (which we can wave off as a curiosity) of "how a finite mind can grasp an infinite object" into a *constitutive ontological problem*.

The upshot of this immense[10] challenge (besides making the cognitive-foundations viable), is a more concrete take on idealizations (and thus their legitimacy and price). Even without committing to any particular position, the value of an orderly, methodological approach to exploring the many related debates (not only in philosophy-of-mathematics, but within mathematics itself), could prove immense. (Lavine, 1994), (Oppy, 2006)

Set Theoretical Foundations?

A foundation for mathematics needs to be rich enough to actually reach, ground, all of mathematics – and *that*, in a sense, set-theory (arguably) does. But that base also needs to be simple (or else, why not just keep mathematics as a whole?). (The promise of) being able to make do with a single theory and a single ontological type (sets) understandably seemed (a century ago) to fair quite well with the aspiration for simplicity. However, following modern understanding of cognition (which's relevancy here was demonstrated in Part I), as well as the developments in logic & in set-theory since then, we now know better than that.

On a different layer: the sets-universe also fails – as a foundation – to answer to mathematicians' introspection. *Mathematical foundations* (in (Shapiro, 2004)'s terminology)

[10] A challenge that is – from the Platonist/Logicist points-of-view – pointless.

are not enough. When a mathematician speaks of the number 2, she *does not* mean the set {{}, {{}}} (not even implicitly, non-consciously – until proven otherwise).

Bibliography

Benacerraf Paul Mathematical Truth [Journal] // The Journal of Philosophy. - 1973. - Vol. 70. - pp. 661-679.

Benjafield John G., Smilek Dan and Kingstone Alan Cognition (Fourth Edition) [Book]. - [s.l.] : Oxford University Press, 2010.

Chalmers David J. A Computational Foundation for the Study of Cognition [Report]. - 1994.

Dehaene Stanislas The Number Sense: How the Mind Creates Mathematics [Book]. - [s.l.] : Oxford University Press, 1999.

Devlin Keith The Math Gene [Book]. - [s.l.] : Basic Books, 2000.

Ferreirós José Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics (2nd edition) [Book]. - [s.l.] : Birkhäuser Basel, 2007.

Giaquinto Marcus Visual Thinking in Mathematics [Book]. - [s.l.] : Oxford University Press, 2007.

Goldin Gerald A. Counting on the metaphorical [Journal] // Nature. - September 2001. - Vol. 413.

Greenwood John D. Understanding the "cognitive revolution" in psychology [Journal] // Journal of the History of the Behavioral Sciences. - 1999. - 1 : Vol. 35. - pp. 1-22.

Hadamard Jacques The Psychology of Invention in the Mathematical Field [Book]. - [s.l.] : Dover Publications, 1954.

Hendricks Vincent F. and Leitgeb (eds.) Hannes Philosophy of Mathematics: 5 Questions, [Book]. - [s.l.] : New York: Automatic Press / VIP, 2006.

Horsten Leon and Starikova Irina Special issue: Mathematical knowledge: Intuition, visualization, and understanding [Journal] // Topoi. - April 2010.

Kreisel Georg Informal rigour and completeness proofs [Book Section] // Problems in the Philosophy of Mathematics / book auth. Lakatos I.. - 1967.

Lakoff George and Núñez Rafael Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being [Book]. - [s.l.] : Basic Books, 2000.

Longo Giuseppe and Viarouge Arnaud Mathematical intuition and the cognitive roots of mathematical concepts [Journal] // Topoi. - 2010. - 1 : Vol. 29. - pp. 15-27.

Longo Giuseppe Mathematical Concepts and Physical Objects [Book Section] // Rediscovering Phenomenology / book auth. Boi L., Kerszberg P. and Patras (eds.) F.. - [s.l.] : Springer, 2007.

Longo Giuseppe Mathematical Intelligence, Infinity and Machines: beyond the Goedelitis [Journal] // Journal of Consciousness Studies. - 1999. - 11-12 : Vol. 6.

Longo Giuseppe The Constructed Objectivity of Mathematics and the Cognitive Subject [Book Section] // Quantum Mechanics, Mathematics, Cognition and Action - Proposals in Epistemology. / book auth. Mugur-Schachter (ed.) M.. - [s.l.] : Kluwer, 2002.

Longo Giuseppe The Reasonable Effectiveness of Mathematics and its Cognitive Roots. [Book Section] // Geometries of Nature, Living Systems and Human Cognition - series in "New Interactions of Mathematics with Natural Sciences and Humanities" / book auth. Boi (ed.) L.. - [s.l.] : World Scientific, 2005.

Madden James Where Mathematics Comes From - Book Review [Journal] // Notices of the AMS. - November 2001. - 10 : Vol. 48.

Piccinini Gualtiero Computationalism [Book Section] // The Oxford Handbook of Philosophy and Cognitive Science. - forthcoming.

Shapiro Stewart Foundations of Mathematics: Metaphysics, Epistemology, Structure [Journal] // The Philosophical Quarterly. - 2004. - 214 : Vol. 54. - pp. 16–37.

Shapiro Stewart Oxford Handbook of Philosophy of Mathematics and Logic [Book]. - [s.l.] : Oxford University Press, 2005.

Shapiro Stewart We Hold These Truths to Be Self-Evident: But What Do We Mean by That? [Journal] // The Review of Symbolic Logic. - 2009. - 1 : Vol. 2. - pp. 175-207.

Steiner Mark Mathematics—Application and Applicability [Book Section] // The Oxford Handbook of Philosophy of Mathematics and Logic / book auth. Shapiro Stewart (ed.). - 2005.

Thurston William P. On proof and progress in mathematics [Journal] // Bulletin of the American Mathematical Society. - 1994. - 2 : Vol. 30. - pp. 161-177.

Voorhees Burton Embodied Mathematics [Journal] // Journal of Consciousness Studies. - 2004. - 9 : Vol. 11. - pp. 83–8.